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EMBEDDING $(p, p-1)$ GRAPHS IN THEIR COMPLEMENTS

BY

DAVID BURNS AND SEYMOUR SCHUSTER

ABSTRACT

A graph G is embeddable in its complement $\bar{G}$ if G is isomorphic with a subgraph of $\bar{G}$. A complete characterization is given of those $(p, p-1)$ graphs which are embeddable in their complements. In particular, let G be a $(p, p-1)$ graph where $p \geqq 6$ if p is even and $p \geqq 9$ if p is odd; then G is embeddable in $\bar{G}$ if and only if G is neither the star $K_{1,p-1}$ nor $K_{1,n} \cup C_3$ with $n \geqq 4$.

1. Terminology and notation

As is quite usual, the vertex-set and edge-set of the graph G will be denoted by $V(G)$ and $E(G)$, respectively. If $|V(G)| = p$ and $|E(G)| = q$, then G is a *(p, q) graph*; also, G is said to be of *order p*. The *complement* $\bar{G}$ of graph G is defined by specifying its vertex-set and edge-set: (i) $V(\bar{G}) = V(G)$ and (ii) $uv \in E(\bar{G})$ if and only if $u, v \in V(G)$ and $uv \notin E(G)$. The *complete graph* K_n is the graph of order n, having every pair of its vertices joined by an edge. Thus, if G is of order n, then the graph with vertex-set $V(G)$ and edge-set $E(G) \cup E(\bar{G})$ is precisely K_n.

Other classes of graphs that require special notation are *stars* and *cycles*. The star $K_{1,n}$ is the $(n+1, n)$ graph having one vertex of degree n and n vertices of degree 1. (Vertices of degree 1 are *endvertices*.) A cycle of length r will be denoted by C_r.

Suppose H and K are graphs having no vertices in common. Then $H \cup K$, the *union* of H and K, is the graph whose vertex-set is $V(H) \cup V(K)$ and whose edge-set is $E(H) \cup E(K)$. The union of m copies of H will be denoted by mH.

Finally, if the graph F is isomorphic with a subgraph of G, then we say that *F is contained in G* or that *F is embeddable in G*, and we write $F \subset G$.

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2. Introduction and preliminary results

The question of whether graphs are contained in their complements has been the subject of several recent investigations. Indeed, Bollobás and Eldridge [1], Sauer and Spencer [3], as well as the current authors [2], proved the following, independently.

THEOREM A. *Every $(p, p-2)$ graph is contained in its complement.*

We shall have occasion to use the following strengthened version of this theorem which is proved in [4].

THEOREM A′. *If G is any labeled $(p, p-2)$ graph, then there exists an isomorphic embedding ϕ of G into $\bar{G}$ such that ϕ has no fixed vertices.*

Clearly, the embeddings of Theorems A and A′ exist for any $(p, p-n)$ graph if $n \geqq 2$.

As for $(p, p-1)$ graphs, it appears that H. Joseph Straight was the first to observe that nearly all trees are contained in their complements. He proved the following:

THEOREM B. *Every non-trivial tree, which is not a star, is contained in its complement.*

We propose to complete the study of embedding $(p, p-1)$ graphs in their complements, thus obtaining Theorem B as a special case. First, we announce the class $\mathfrak{F}$ of forbidden graphs: $K_1 \cup C_3$, $K_1 \cup C_4$, $K_1 \cup 2C_3$, $K_{1,1} \cup C_3$, $K_{1,p-1}$, and $K_{1,n} \cup C_3$ where $n \geqq 4$. (See Fig. 1.) Now, we begin by disposing of a most bothersome class of $(p, p-1)$ graphs.

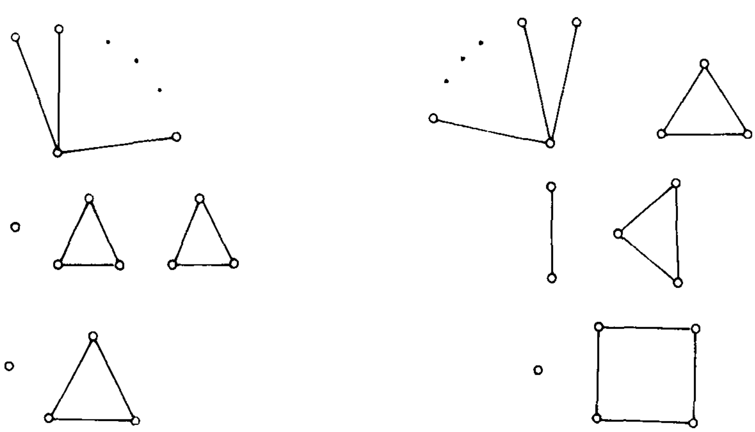

Fig. 1. The class $\mathfrak{F}$ of forbidden graphs.

THEOREM 1. *Let T be any tree, trivial or non-trivial. If G is the union of T and m $(\geqq 1)$ disjoint cycles and $G \notin \mathfrak{F}$, then $G \subset \bar{G}$.*

PROOF. Suppose that among the m cycles of G, there is a cycle C_r with $r \geqq 5$. Let $V(C_r) = \{v_1, v_2, \cdots, v_r\}$ and consider the $(k, k-2)$ graph $G_1 = G - \{v_1, v_2, \cdots, v_{r-1}\}$. By Theorem A′ we know that there exists an isomorphic embedding α of G_1 into $\bar{G}_1$ such that α has no fixed vertices. Further, by Theorem B, we know that there is an isomorphic embedding β of $C_r - v_r$ into $\overline{C_r - v_r}$. The union of mappings α and β provides an isomorphic embedding ϕ of G into $\bar{G}$, as follows:

$$\phi(v) = \alpha(v) \qquad \text{for} \quad v \in V(G_1),$$

$$\phi(v_i) = \beta(v_i) \qquad \text{for} \quad 1 \leqq i \leqq r-1.$$

Since $\alpha(v_r) \neq v_r$, the edges $\phi(v_{r-1})\phi(v_r)$ and $\phi(v_1)\beta(v_r)$ are in $\bar{G}$. Certainly, all the other edges of $\phi(G)$ are in $\bar{G}$, so ϕ is the desired isomorphic embedding.

Having proved the theorem for graphs with a cycle C_r, with $r \geqq 5$, as one of its components, we assume henceforth that the m cycles of G are 3-cycles or 4-cycles. The remainder of the proof proceeds by induction on m.

If $m = 1$, then $G = T \cup C_3$ or $G = T \cup C_4$.

Consider $G = T \cup C_3$. Since $G \notin \mathfrak{F}$, T is of order at least 3. If T is a star, then $G = K_{1,2} \cup C_3$ or $G = K_{1,3} \cup C_3$; in each case, it is easily verified that $G \subset \bar{G}$. If T is not a star, then T is of order $t \geqq 4$. For $t = 4, 5$ the embeddings of G into $\bar{G}$ are shown in Fig. 2, where solid lines indicate edges of G and dashed lines indicate edges of $\bar{G}$. If $t \geqq 6$, then there are endvertices $x, y \in V(T)$ such that $T - \{x, y\}$ is not a star; hence, there is an isomorphic mapping $\alpha: T - \{x, y\} \to \overline{T - \{x, y\}}$. Calling $V(C_3) = \{u_1, u_2, u_3\}$, we define the mapping ϕ as follows:

$$\phi(u_1) = x, \quad \phi(u_2) = y, \quad \phi(u_3) = u_3, \quad \phi(x) = u_1,$$

$$\phi(y) = u_2 \quad \text{and} \quad \phi(v) = \alpha(v)$$

$$\text{for} \quad v \in V(T - \{x, y\}).$$

Then ϕ is an embedding of $T \cup C_3$ into $\overline{T \cup C_3}$.

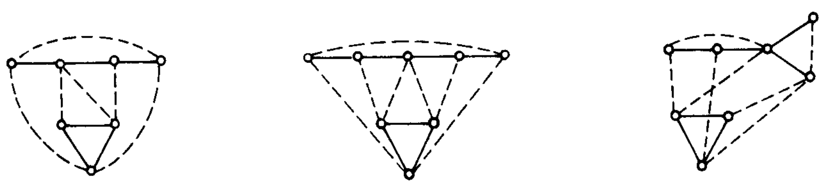

Fig. 2.

Consider $G = T \cup C_4$. Since $G \not\in \mathfrak{F}$, T is non-trivial. If T is a star, it is easy to see that $G \subset \bar{G}$. If T is not a star, then let α be an embedding of T into $\bar{T}$. Also, let $v_1, v_2 \in V(T)$ and $V(C_4) = \{u_1, u_2, u_3, u_4\}$. We define ϕ as follows:

$$\phi(u_1) = u_1, \quad \phi(u_2) = \alpha(v_1), \quad \phi(u_3) = u_3, \quad \phi(u_4) = \alpha(v_2),$$

$$\phi(v_1) = u_2, \quad \phi(v_2) = u_4, \quad \text{and} \quad \phi(v) = \alpha(v)$$

$$\text{for} \quad v \in V(T) \quad \text{and} \quad v \neq v_1, v_2.$$

Then, ϕ is an embedding of G into $\bar{G}$ as indicated in Fig. 3. This completes the argument for $m = 1$.

Assume, now, that $G \subset \bar{G}$ for any graph G satisfying the hypothesis of the Theorem, where $m < k$ and $k \geqq 2$. Let H be a $(p, p-1)$ graph satisfying the hypothesis, where H is the union of a tree and k cycles C_r, with $r = 3$ or 4.

We consider two cases depending on whether C_4 is a component of H.

Case 1. Assume one component of H to be a C_4. Then H has another cycle as a component. Consider $H_1 = H - \{C_4, C_r\}$, with $r = 3$ or 4.

Before we may apply the induction hypothesis to H_1, we must tediously eliminate those cases in which $H_1 \in \mathfrak{F}$ or H_1 degenerates to become K_1.

If $H_1 = K_1$, then $H = K_1 \cup C_4 \cup C_3$ or $H = K_1 \cup 2C_4$; in both cases, the embeddings of H into $\bar{H}$ are simple to construct. If $H_1 = K_1 \cup C_3$, then $H = K_1 \cup C_4 \cup 2C_3$ or $H = K_1 \cup 2C_4 \cup C_3$; again, it is easily verified that $H \subset \bar{H}$.

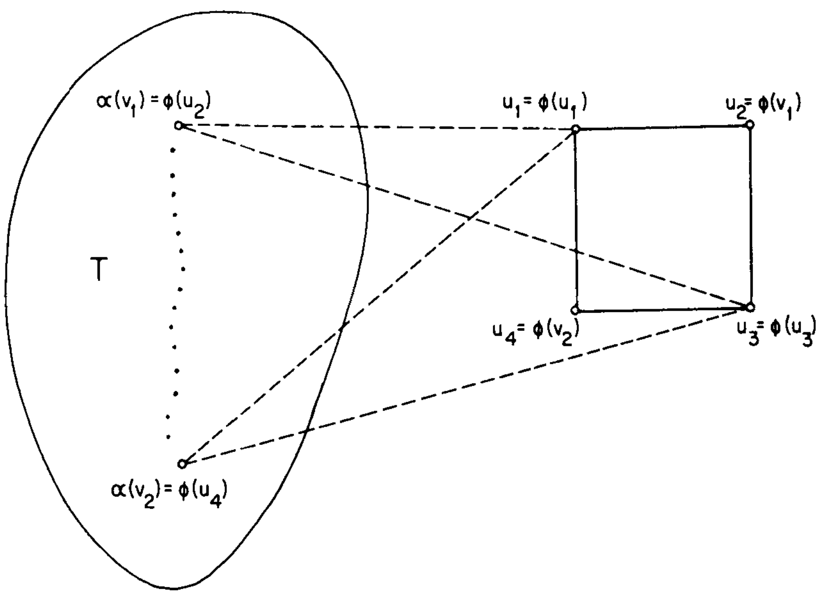

Fig. 3.

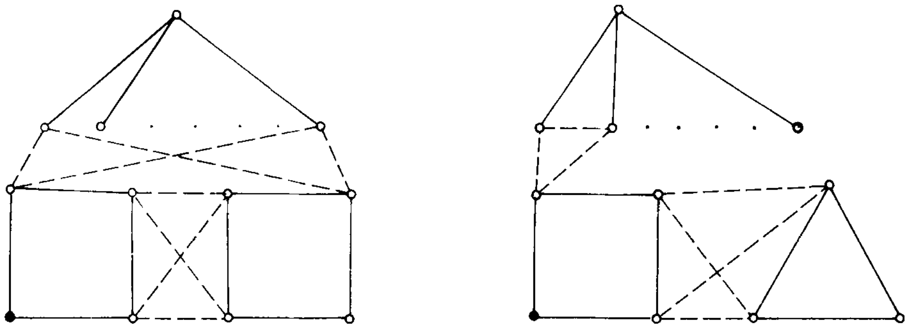

Fig. 4.

If H_1 is a star, then the embeddings of H into $\bar{H}$ are exhibited in Fig. 4, where a solid vertex indicates the presence of dashed edges between that vertex and all other vertices in the diagram which are not already incident with dashed edges. If $H_1 = K_{1,1} \cup C_3$, then $H = K_{1,1} \cup 2C_4 \cup C_3$ or $H = K_{1,1} \cup C_4 \cup 2C_3$; if $H_1 = K_1 \cup C_4$, then $H = K_1 \cup 3C_4$ or $H = K_1 \cup 2C_4 \cup C_3$; and if $H_1 = K_1 \cup 2C_3$, then $H = K_1 \cup 2C_4 \cup 2C_3$ or $H_1 = K_1 \cup C_4 \cup 3C_3$. In all these cases, it is a simple matter to verify that $H \subset \bar{H}$.

Finally, if $H_1 = K_{1,n} \cup C_3$, with $n \geqq 4$, $H = K_{1,n} \cup 2C_4 \cup C_3$ or $H = K_{1,n} \cup 2C_3 \cup C_4$. In the latter case, the embedding of H in $\bar{H}$ is obvious. In the former case, we consider the graph $H_2 = H - \{C_4, C_3\} = K_{1,n} \cup C_4$, in which case we may apply the induction hypothesis to H_2; since $H_2 \subset \bar{H}_2$ and $C_4 \cup C_3 \subset \overline{C_4 \cup C_3}$, we have $H \subset \bar{H}$.

In all the remaining cases, H_1 satisfies the induction hypothesis, so $H_1 \subset \overline{H_1}$. Since $C_4 \cup C_r \subset \overline{C_4 \cup C_r}$ for any $r \geqq 3$, we conclude that $H \subset \bar{H}$.

Case 2. Assume that every cyclic component of H is a 3-cycle; i.e., $H = T \cup kC_3$.

Suppose that $T = K_1$. Then $k > 2$, otherwise $H \in \mathfrak{F}$. Since $kC_3 \subset \overline{kC_3}$ for $k > 2$, we have $H \subset \bar{H}$.

Suppose T is the star $K_{1,n}$. If $n = 1$, the embedding of $H = K_{1,1} \cup kC_3$ in its complement is obvious for $k = 2$, 3, and 4. For $n > 1$ and $k = 2$, 3, and 4, we exhibit the embeddings in Fig. 5. If $k > 4$, the graph $H_2 = H - 3C_3$ obeys the induction hypothesis; hence, $H_2 \subset \bar{H}_2$. And, since $3C_3 \subset \overline{3C_3}$, we have $H \subset \bar{H}$. This completes the analysis for T a star.

We now turn to the general case in which T is neither K_1 nor a star $K_{1,n}$. Let $v_1, v_2 \in V(T)$ and $\alpha: T \to \bar{T}$ be an isomorphic embedding in which $\alpha(v_i) = v_1$ and $\alpha(v_j) = v_2$. If $k = 2$, we call the 3-cycles u_1, u_2, u_3 and w_1, w_2, w_3, and then define ϕ as follows (see Fig. 6):

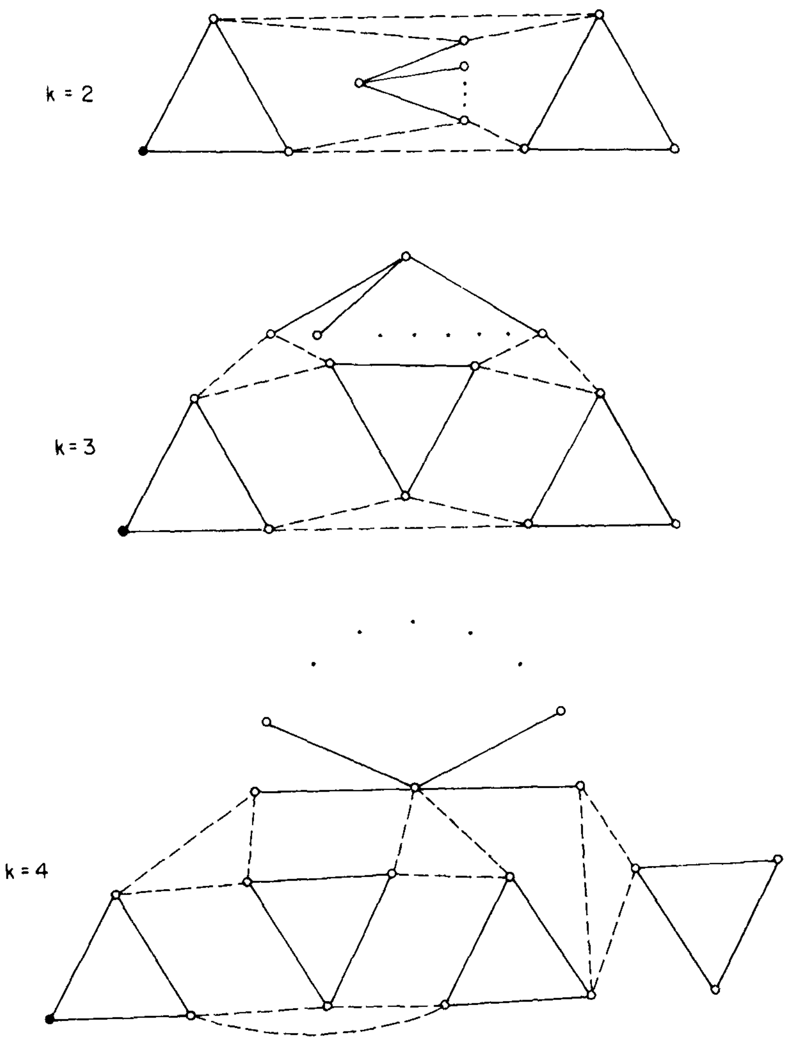

Fig. 5.

$$\phi(u_1)=u_1,\quad \phi(u_2)=w_2,\quad \phi(u_3)=v_1,$$

$$\phi(w_1)=w_1,\quad \phi(w_2)=u_2,\quad \phi(w_3)=v_2,$$

$$\phi(v_i)=u_3,\quad \phi(v_j)=w_3,\quad \text{and}\quad \phi(v)=\alpha(v)$$

for the remaining $v \in V(T)$. The mapping ϕ is an isomorphic embedding of $T \cup 2C_3$ into its complement. If $k=3$, then $3C_3 \subset \overline{3C_3}$ coupled with $T \subset \bar{T}$ implies that $H = T \cup 3C_3$ is contained in its complement. For $k>3$, $H_3 = H - 3C_3$ satisfies the induction hypothesis; hence, $H_3 \subset \overline{H_3}$ and $3C_3 \subset \overline{3C_3}$ completes the proof that $H \subset \bar{H}$.

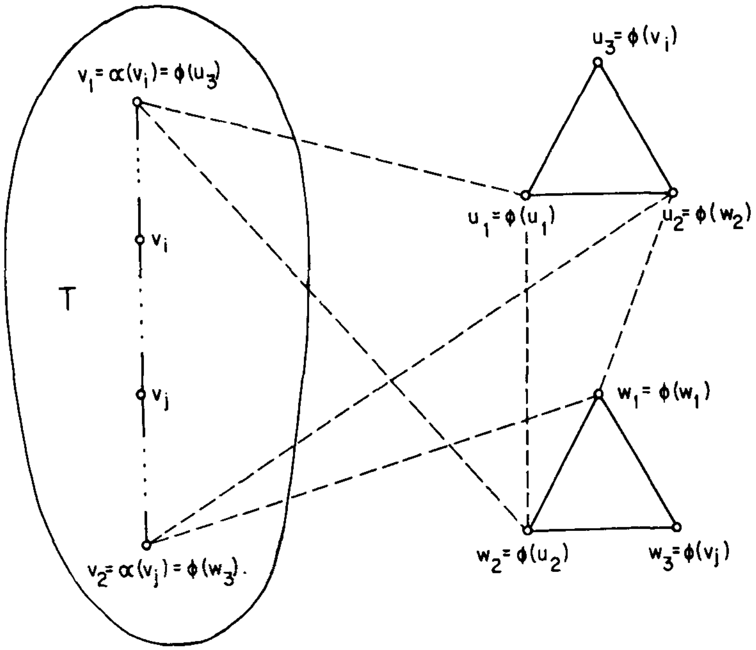

Fig. 6.

3. Principal theorem

We are now able to present the complete classification of those $(p, p-1)$ graphs which are contained in their complements.

THEOREM 2. *Let G be any $(p, p-1)$ graph, with $p \geqq 3$. Then G is contained in its complement if and only if $G \notin \mathfrak{F}$.*

PROOF. Clearly, if $G \in \mathfrak{F}$ then G cannot be isomorphically embedded in $\bar{G}$. We proceed with the converse.

Our attention will be restricted to the case in which G is disconnected, for if G is connected it is a tree in which case Theorem B applies.

If v is an isolated vertex of G, then $G - v$ is a $(p-1, p-1)$ graph. Hence, $G - v$ is either a union of cycles or else it contains a vertex u of degree $\geqq 3$. The former case is covered by Theorem 1. In the latter case, $G - \{v, u\}$ is a $(p-2, p-k)$ graph with $k \geqq 4$. Thus, by the remark following Theorem A′, we know that there is an isomorphic embedding $\phi\colon G - \{v, u\} \to \overline{G - \{v, u\}}$. Defining $\phi(v) = u$ and $\phi(u) = v$ provides an embedding of G in $\bar{G}$.

If G possesses no isolated vertices, then it must have a tree T of order $t \geqq 2$ as one of its components (for every cyclic component with n vertices has at least n edges). Then $G - T$ is a $(p-t, p-t)$ graph. Either $G - T$ is a disjoint union of cycles or $G - T$ contains a vertex w whose degree is at least 3. The former alternative, in which G is the union of a tree and cycles, is covered by Theorem 1. In the second alternative, $G - T - w$ is a $(p - t - 1, p - t - s)$ graph with $s \geqq 3$; hence, there is an isomorphic embedding α of $G - T - w$ into its complement. Also, if z is a vertex of maximal degree in T, then there is an isomorphic embedding $\beta: T - z \to \overline{T - z}$, because $T - z$ is either a $(t - 1, t - n)$ graph with $n \geqq 3$ or it is a K_1. By defining

$$\phi(e) = z, \qquad \phi(z) = w,$$

$$\phi(x) = \alpha(x) \quad \text{for} \quad x \in V(G - T - w),$$

$$\phi(y) = \beta(y) \quad \text{for} \quad y \in V(T - z),$$

we obtain an isomorphic embedding ϕ of G into $\bar{G}$.

This completes the proof of our classification Theorem.

Focusing primarily on the two infinite classes of forbidden graphs, we obtain the following.

COROLLARY. *Let G be any $(p, p - 1)$ graph where $p \geqq 6$ if p is even and $p \geqq 9$ if p is odd. Then $G \subset \bar{G}$ if and only if G is neither the star $K_{1,p-1}$ nor $K_{1,n} \cup C_3$ for $n \geqq 4$.*

ACKNOWLEDGEMENT

We wish to thank Dick Schelp for observing that a rearrangement of the cases in an earlier version of this paper greatly simplified the proof of Theorem 2. Indeed, following Schelp's suggestion enabled us to eliminate an induction argument, the anchor of which required the verification that $G \subset \bar{G}$ for 343 (9, 8) graphs!

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WESTERN MICHIGAN UNIVERSITY
KALAMAZOO, MICHIGAN 49008 USA
AND
CARLETON COLLEGE
NORTHFIELD, MINNESOTA 55057 USA